Integration

Definition. *Given a function $f : [a, b] \to \mathbb{R}$ we define the* ***signed area*** *under the graph of f to be*

$$A_{above} - A_{below}$$

where A_{above} is the area enclosed between the x-axis and the sections of the graph of f lying above the x-axis, and A_{below} is the area enclosed between the x-axis and the sections of the graph of f lying below the x-axis.

We cannot consistently assign a value to the signed area under the graph of every function. The graphs of some functions are simply to "wild" to admit a sensible notion of area. We will refer to those functions which do as **Riemann integrable** and denote the signed area under the graph of f over the interval $[a, b]$ by

$$\int_a^b f(x)\,dx.$$

An example of a function which is not Riemann integrable is the characteristic function on the rationals:

$$f(x) := \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Most functions that you are familiar with are Riemann integrable. For example, see the theorem below.

Theorem. *If $f : [a, b] \to \mathbb{R}$ is continuous, then it is Riemann integrable.*

We will not develop the theory necessary to characterise exactly which functions are Riemann integrable here.

An expression of the form

$$\int_a^b f(x)\,dx$$

is referred to as a **definite integral**. Whenever an integral is written with terminals it denotes a definite integral, that is, it's a numerical value for the signed area under the graph of f. This is in contrast to an expression of the form

$$\int f(x)\,dx$$

which is referred to as an **indefinite integral**, and is simply notation for the general antiderivative of f. The justification for why the notation is chosen to be so similar for two a priori unrelated concepts is provided by the Fundamental Theorem of Calculus below.

Note that we adopt the following conventions:

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

and

$$\int_a^a f(x)\,dx = 0.$$

Theorem (Properties of the Definite Integral). *Suppose $f : [a, b] \to \mathbb{R}$ is a Riemann integrable function.*

(i) If $k \in \mathbb{R}$, then kf is Riemann integrable and

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$$

(ii) If $g : [a, b] \to \mathbb{R}$ is another Riemann integrable function, then $f + g$ is also Riemann integrable and

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

(iii) If $a < c < b$, then f is Riemann integrable on $[a, c]$ and $[c, b]$ and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

(iv) If $g : [a, b] \to \mathbb{R}$ is another Riemann integrable function and $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

(v) The function $|f| : [a, b] \to \mathbb{R}$ is also Riemann integrable, and

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b \big|f(x)\big|\,dx.$$

Theorem (Fundamental Theorem of Calculus). *Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function and $c \in [a, b]$. Define a new function F with the same domain*

$$F(x) = \int_c^x f(t)\,dt.$$

Then, $F(x)$ is differentiable and $F'(x) = f(x)$. Moreover, if F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The Fundamental Theorem of Calculus implies that we can compute the signed area under the graph of f using *any* antiderivative of f. However, antidifferentiation is simply the inverse process of differentiation which, a priori, has nothing to do with computing areas! The Fundamental Theorem of Calculus gives an amazing connection between the concept of differentiation (involving computing the gradients of tangent lines to a graph) and integration (involving finding the signed area under a graph). This is the tremendous utility of our theorem, that given f, we can use any of its antiderivatives to efficiently compute definite integrals.

Theorem (Mean Value Theorem for Integrals). *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists $x_0 \in [a, b]$ such that*

$$\int_a^b f(x)\,dx = f(x_0)(b - a).$$

This quantity $f(x_0)$ is called the mean value of f over the interval $[a, b]$, and is denoted $\bar{f}$. Hence,

$$\bar{f} = \frac{1}{b - a} \int_a^b f(x)\,dx.$$

Given a differentiable function expressed as an algebraic combination of elementary functions (i.e. polynomials, trigonometric functions, exponentials etc.) you can use the basic differentiation rules to compute the derivative. It may become arbitrarily complicated and take a long time, but nonetheless the computation is possible and the result will itself be expressed in terms of elementary functions.

In contrast, suppose we have a function expressed in terms of elementary functions, and suppose it is continuous, so we are guaranteed that the antiderivative exists. It is not necessarily possible to express the antiderivative in terms of elementary functions! Even though the antiderivative exists, it is not necessarily possible to write down a formula for it. Examples of functions whose antiderivatives are not expressible include many seemingly innocuous functions i.e. e^{-x^2}, $\sin(x^2)$ etc. Therefore, it is easy to exhibit functions which are impossible to analytically integrate. For such functions we usually invent some notation for their antiderivative, and employ numerical methods to compute its values.

However, due to the Fundamental Theorem of Calculus it is of great importance to be able to find antiderivatives for functions whenever possible. To this end there are four primary techniques for analytically determining antiderivatives:

(i) integration by parts,
(ii) integration by substitution,
(iii) integration using trigonometric identities,
(iv) integration using partial fractions.

The first two of these methods are the antiderivative equivalents of the product rule and the chain rule, respectively.

Integration by Parts.

Theorem (Integration by Parts for Definite Integrals). *Suppose $f, g : [a, b] \to \mathbb{R}$ are both continuously differentiable functions. Then,*

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx,$$

Theorem (Integration by Parts for Indefinite Integrals). *Suppose $f, g : [a, b] \to \mathbb{R}$ are both continuously differentiable functions. Then,*

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx,$$

Integration by Substitution.

Theorem (Substitution Rule for Definite Integrals). *Suppose f is a continuous real-valued function defined on the image of a continuously differentiable function $g : [a, b] \to \mathbb{R}$.*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Theorem (Substitution Rule for Indefinite Integrals). *Suppose f is a continuous real-valued function defined on the image of a continuously differentiable function $g : [a, b] \to \mathbb{R}$. Letting $u = g(x)$ we have,*

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

The theorem above justifies the following procedure for evaluating indefinite integrals.

(1) Let $u = g(x)$.
(2) Replace $g'(x)\,dx$ with du and express the integral purely in terms of u.
(3) Integrate the corresponding expression in the integral with respect to u.
(4) Replace u with $g(x)$.

Integration using Trigonometric Identities.
We will not go into detail explaining how to use trigonometric identities to evaluate integrals here, beyond mentioning that powers of sin and cos can be "reduced" to lower powers using the well known identities:

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$
$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

Integration using Partial Fractions.
When we seek to integrate **rational functions**, that is, functions of the form $g(x)/h(x)$ where g and h are polynomials, the following procedure is often useful:

(1) If the numerator has degree greater than or equal to the denominator, then first divide the polynomials.
(2) Factorize the numerator and denominator of the resulting proper fraction.
(3) Split the proper fraction into partial fractions using the rules below.
(4) Integrate each term separately.

The rules we employ to split proper fractions into partial fractions are as follows:

(i) For each distinct linear factor $(x - a)$ in the denominator we obtain an expression of the form

$$\frac{A}{x - a}$$

where $A \in \mathbb{R}$.

(ii) For each repeated linear factor $(x - a)^n$ in the denominator we obtain an expression of the form

$$\frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \cdots + \frac{A_n}{(x - a)^n}$$

where $A_1, A_2, \cdots, A_n \in \mathbb{R}$.

(iii) For each irreducible quadratic factor $ax^2 + bx + c$ in the denominator we obtain an expression of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

where $A, B \in \mathbb{R}$. Note that a quadratic is said to be **irreducible** if it cannot be factorized into linear factors over $\mathbb{R}$.

Since any polynomial with real coefficients can be factorized into linear factors and irreducible quadratics, these rules cover every possibility.

Applications of Integration.

DEFINITION. *The volume of the solid obtained by rotating about the x-axis the region under the curve $y = f(x)$ from $x = a$ to $x = b$ can be found by integrating the cross-sectional area as a function of x*

$$V = \int_a^b \pi\Big(f(x)\Big)^2 \, dx$$

This volume can also be computed via the formula

$$V = \int_a^b 2\pi x f(x) \, dx.$$

Similar formulas hold for rotations about the y-axis.

DEFINITION. *The surface area of the solid obtained by rotating about the x-axis the region under the curve $y = f(x)$ from $x = a$ to $x = b$ can be computed via the formula*

$$A = \int_a^b 2\pi f(x)\sqrt{1 + \Big(f'(x)\Big)^2} \, dx.$$

A similar formula holds for rotations about the y-axis.

DEFINITION. *The length of a curve $y = f(x)$ from $x = a$ to $x = b$ can be computed via the formula*

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx.$$

If the curve is described parametrically $\gamma(t) = (x(t), y(t))$ for $\alpha \leq t \leq \beta$, then the length can be computed via the formula

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.$$