Combinatorics

Counting Techniques.

(i) The Addition Principle. Assume that there are n_1 ways for event E_1 to occur, n_2 ways for event E_2 to occur, …, and n_k ways for event E_k to occur. If the ways for these different events to occur are pairwise disjoint, then the number of ways for at least one of the events $E_1, E_2, \ldots, E_k$ to occur is

$$n_1 + n_2 + \cdots + n_k = \sum_{i=1}^{k} n_i.$$

(ii) The Multiplication Principle. Assume that an event E can de decomposed into k ordered events $E_1, E_2, \ldots, E_k$, and that there are n_1 ways for event E_1 to occur, n_2 ways for event E_2 to occur,...,n_k ways for event E_k to occur. Then the total number of ways for the event E to occur is given by:

$$n_1 \times n_2 \times \cdots \times n_k = \prod_{i=1}^{k} n_i.$$

(iii) Ordered sampling without replacement (permutations). Suppose we have n distinct objects, and we wish to arrange r of them in a row. The number of such possibilities is given by

$$^nP_r = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

In particular, when $r = n$ we obtain $n!$.

(iv) Ordered sampling with replacement. Suppose we have n distinct objects, and we wish to arrange r of them in a row, where we allow repetitions. The number of such possibilities is given by

$$n^r.$$

(v) Unordered sampling without replacement (combinations). Suppose we have n distinct objects, from which we wish to choose a collection of size r. The number of such possibilities is given by

$$\binom{n}{r} = {}^nC_r = \frac{^nP_r}{r!} = \frac{n!}{r!(n-r)!}$$

(vi) Unordered sampling with replacement. Suppose we have n distinct objects, from which we wish to choose a collection of size r, where we allow repetitions. The number of such possibilities is given by

$$\binom{r+n-1}{r}$$

Pascal's Triangle. Pascal's Triangle is a triangular array of natural numbers. The rows are conventionally labelled so that the top row is the 0^{th} row, and the left-most entry on any given row is the 0^{th} entry. The 0^{th} entry of the 0^{th} row is 1. The following are all equivalent definitions of the k^{th} entry of the n^{th} row (where $k \geq 0,\ n > 0$) which is denoted $\binom{n}{k}$:

(i) the number of shortest paths from this entry to the top of the triangle.
(ii) the coefficient of the k^{th} term in the expansion of $(x+y)^n$.
(iii) inductively using the formula $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.
(iv) the number of ways of choosing k objects from n objects: $\dfrac{n!}{k!(n-k)!}$.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$$\binom{0}{0}$$
$$\binom{1}{0}\ \binom{1}{1}$$
$$\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}$$
$$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}$$
$$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$$
$$\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5}$$

The following important theorem codifies the equivalence of (ii) and (iv) above.

Theorem (Binomial Theorem). *Let $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$. Then,*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Combinatorial Identities. Combinatorial identities can often be proved using three different techniques:

(i) committee-forming arguments
(ii) brute algebra
(iii) path-counting on Pascal's Triangle.

We only list three such identities here, but there are many more.

Theorem. *Assume n and r are non-negative integers with $0 \leq r \leq n$.*

(i) $\binom{n}{r} = \binom{n}{n-r}$.

(ii) $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$.

(iii) $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$.

The Principle of Inclusion/Exclusion. The principle of inclusion/exclusion is used to count the number of items that are in the union of two or more sets. This can be thought of as counting items that have "at least one" of a number of properties. For example, suppose we have finite sets A, B, C, then the number of elements in A or B or C is

$$\#(A \cup B \cup C) = \#(A) + \#(B) + \#(C) - \#(A \cap B) - \#(A \cap B) - \#(B \cap C) + \#(A \cap B \cap C).$$

When applying the principle it is important to consider how many times each item is counted, instead of memorizing formulas.

Pigeonhole Principle. The pigeonhole principle states that if we distribute n items into k boxes where $k < n$, then at least one box must contain more than one item. This can obviously be generalized.

Further Identities. Below we list several further identities which often prove useful in calculations.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$