Polynomials

Definition. *We define $\mathbb{R}[x]$ to be the set of all polynomials with indeterminate x and coefficients in $\mathbb{R}$. That is, $\mathbb{R}[x]$ is the set of all expressions of the form*

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$$

*where $a_i \in \mathbb{R}$. We can perform all the usual algebraic operations on these polynomials in the obvious way. The highest value of i for which a_i is non-zero is said to be the **degree** of the polynomial $p(x)$. A polynomial whose leading term has a coefficient of 1 is called a **monic polynomial**. A constant $d \in \mathbb{R}$ such that $p(d) = 0$ is referred to as a **root** or **zero** of the polynomial $p(x)$, or a **solution** of the equation $p(x) = 0$.*

Theorem. *Suppose $f(x)$ and $g(x)$ are polynomials.*

(i) $\deg\Big(f(x)g(x)\Big) = \deg(f(x)) + \deg(g(x))$

(ii) $\deg\Big(f(x) + g(x)\Big) \leq \max\Big(\deg(f(x)), \deg(g(x)\Big)$

(iii) $\deg\Big(f(g(x))\Big) = \deg\Big(g(f(x))\Big) = \deg(f(x))\deg(g(x))$

Note that there exist formulas which relate the coefficients of a polynomial to its roots. For example, suppose the quadratic $ax^2 + bx + c$ has roots α_1, α_2, then we can write

$$ax^2 + bx + c = a(x - \alpha_1)(x - \alpha_2)$$

and expanding we obtain

$$\alpha_1 + \alpha_2 = -\frac{b}{a}, \quad \alpha_1\alpha_2 = \frac{c}{a}.$$

There are similar formulas for higher degree polynomials. They are called **Vieta's formulas**.

Theorem (Division Algorithm). *For any polynomials $p(x), d(x) \in \mathbb{R}[x]$ where $d(x) \neq 0$, there exist unique polynomials $q(x), r(x) \in \mathbb{R}[x]$ such that*

$$p(x) = q(x)d(x) + r(x),$$

where $\deg(r(x)) < \deg(d(x))$.

Theorem (Remainder Theorem I). *Suppose $p(x) \in \mathbb{R}[x]$ is any non-zero polynomial, and $\alpha \in \mathbb{R}$. Then there exists a polynomial $q(x) \in \mathbb{R}[x]$ such that*

$$p(x) = q(x)(x - \alpha) + p(\alpha).$$

Corollary (Factor Theorem). *Suppose $p(x) \in \mathbb{R}[x]$ is any non-zero polynomial, and $\alpha \in \mathbb{R}$. Then $p(\alpha) = 0$ iff $(x - \alpha)$ is a factor of $p(x)$.*

Corollary. *Suppose $p(x) \in \mathbb{R}[x]$ is any non-zero polynomial of degree n. Then $p(x)$ has at most n distinct zeros in $\mathbb{R}$.*

Theorem. *Suppose $p(x), q(x) \in \mathbb{R}[x]$ are two polynomials of degree at most n. If $p(x) = q(x)$ for $n + 1$ values of x, then the polynomials $p(x)$ and $q(x)$ are equal.*

Theorem (Remainder Theorem II). *Suppose $p(x) \in \mathbb{R}[x]$ is any non-zero polynomial, and $\alpha, \beta \in \mathbb{R}$. Then there exists a polynomial $q(x) \in \mathbb{R}[x]$ such that*

$$p(x) = q(x)(\beta x - \alpha) + p(\frac{\alpha}{\beta}).$$

Theorem (Rational Root Theorem). *Suppose $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial whose coefficients are all integers. Let α, β be coprime integers. If $p\big(\frac{\alpha}{\beta}\big) = 0$, then β divides a_n, and α divides a_0.*