Differential Equations

Definition. *A **differential equation** is an equation that relates one or more functions of one or more independent variables with their derivatives. If the equation only involves derivatives with respect to one independent variable, then it is said to be an **ordinary differential equation** (ODE). If the equation involves derivatives with respect to two or more independent variables, then it is said to be a **partial differential equation** (PDE). We will only consider ODEs here. The **order** of a differential equation is the order of the highest derivative occurring in the equation. A differential equation of the form*

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

*is said to **linear**. Any differential equation which is not linear is said to be **non-linear**. Typically non-linear equations are much harder to solve. If $f(x) \equiv 0$ above, then we refer to such an equation as a **homogeneous** linear differential equation.*

*Given a differential equation of order n, a function defined on some interval I possessing n continuous derivatives is said to be a **particular solution** of the equation on the interval I. The set of all solutions is referred to as the **general solution**.*

There are several theorems guaranteeing existence and uniqueness of solutions under broad hypotheses. However, these results do not imply that we can easily find a solution, even if a solution is known to exist. In order to actually find a solution there are various techniques that can be employed when the equation has a certain structure (some of which we describe below).

First-order Differential Equations. Two common approaches for finding solutions to first-order differential equations are listed below.

(i) **Separation of variables** may be applied to solve (possibly non-linear) equations of the form

$$y' = f(x)g(y).$$

(ii) **Integrating factors** may be applied to solve equations of the form

$$y' + p(x)y = f(x).$$

Specifically we obtain

$$y = \frac{1}{I(x)}\int I(x)f(x)\,dx$$

where $I(x) = \exp\left(\int p(x)\,dx\right)$. Note that, in principal, integrating factors allow you to solve any linear first-order equation.

Second-order Differential Equations. Unlike first-order differential equations, there does not exist a general method for explicitly solving all second-order differential equations. Hence, we will restrict ourselves to those equations which are linear with constant coefficients $a, b, c \in \mathbb{R}$,

$$ay'' + by' + c = f(x).$$

The general procedure for such equations is as follows:

(1) Find the general solution of the associated homogeneous equation

$$ay'' + by' + c = 0.$$

(2) Find a particular solution to the original equation.
(3) Add the general solution of the homogeneous equation to the particular solution to obtain the general solution to the original equation.

The first step above can be achieved by considering the roots $\lambda_1, \lambda_2 \in \mathbb{C}$ of the characteristic polynomial

$$a\lambda^2 + b\lambda + c = 0.$$

There are three possible cases based on the sign of the discriminant $\Delta = b^2 - 4ac$.

(i) If $\Delta > 0$, then we have distinct real roots $\lambda_1 \neq \lambda_2$, and the homogeneous solution is

$$y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$

(ii) If $\Delta < 0$, then we have distinct (proper) complex roots (in this case they are complex conjugates $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$), and the homogeneous solution is

$$y(x) = e^{\alpha x}\Big(A\cos(\beta x) + B\sin(\beta x)\Big)$$

(iii) If $\Delta = 0$, then we have equal roots $\lambda_1 = \lambda_2 =: \lambda$, and the homogeneous solution is

$$y(x) = (A + Bx)e^{\lambda x}$$