FUNCTIONS

DEFINITION. *Let X and Y be sets. A* ***function*** *from X to Y is a rule which assigns to each element of X an element of Y. The set X is referred to as the* ***domain*** *of the function. The set Y is referred to as the* ***codomain*** *of the function. Often X and Y will be subsets of the real numbers, but this need not be the case.*

We usually employ the notation $f : X \to Y$ where f is function with domain X and codomain Y. We often denote the elements of the domain by $x \in X$, and write $f(x) \in Y$ to denote the element to which x is assigned. This is referred to as the value of f at x, or the **image** of x under f. This is sometimes written

$$x \mapsto f(x).$$

We also refer to the element x which is being mapped as the argument of the function. Note that $f(x)$ is only defined when x is in the domain of f. You can use any letter or symbol to denote a function and its argument, however, what we have described above is often used.

EXAMPLES:

(i) The function $f : [-2, 19] \to \mathbb{R}$, $f(x) = x^2$ assigns to each real number in the given interval the square of that real number.

(ii) The function $g : \mathbb{R}\backslash\{-1, 1\} \to \mathbb{R}$,

$$g(t) = \frac{t^3 + 2t}{t^2 - 1}$$

assigns to each $t \in \mathbb{R}\backslash\{-1, 1\}$ the number $\frac{t^3+2t}{t^2-1}$.

(iii) The function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

assigns 0 to each rational number, and 1 to each irrational number.

(iv) The rule which assigns to each $x \in \mathbb{R}$ the number of 7s in its decimal expansion (if this is finite), and assigns the number π to each x which has infinite number of 7s in its decimal expansion, is a valid function from $\mathbb{R}$ to $\mathbb{R}$.

(v) Any (non-vertical) line $y = mx + c$ can be considered as a function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = mx + c$.

(vi) The rule $p(n) =$ "the n^{th} prime number", is a valid function $p : \mathbb{N} \to \mathbb{R}$.

(vii) Addition can be thought of as a function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. To each pair of real numbers $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$ we assign the real number $x_1 + x_2$. We write this as $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,

$$f((x_1, x_2)) = x_1 + x_2.$$

(viii) The **identity function** can be defined on any set X. It is written $f : X \to X$, $f(x) = x$. It outputs exactly what is input into it.

The point of the above examples is to make clear that a function can have any rule whatsoever. The rule does not have to be given by some algebraic expression. It does not have to be given by one uniform condition. It does not have to be computable in any practical sense. The only thing that is important is that each element in X is assigned to precisely one element of Y.

If we have a function f with domain $X = \mathbb{R}$ and codomain $Y = \mathbb{R}$, then we can plot on a Cartesian plane the points $(x, f(x))$ for each $x \in X$. When we do this for all $x \in X$ we obtain the **graph** of f. It provides a pictorial representation of f.

DEFINITION. *If we are given an algebraic rule for a function without the domain being explicitly mentioned, then we define the* ***implied domain*** *or* ***maximal domain*** *to be the largest set of real numbers for which the function is well-defined.*

For example, the implied domain for $f(x) = \sqrt{x}$ is $[0, \infty)$.

You should think of a function as being like a machine. It has a collection of inputs (the set of all inputs is called the domain), and for each input it produces exactly one output (the set containing the outputs is called the codomain).

DEFINITION. *For a function $f : X \to Y$ the set of all outputs which are actually produced is called the* ***range****. That is,*

$$\text{range}(f) = \{y \in Y : f(x) = y \text{ for some } x \in X\}.$$

Note that range(f) is a subset of Y.

DEFINITION. *If the range of a function g lies within the domain of a function f, then we can form the* ***composition function*** *$f \circ g$. This maps elements x in the domain of g as follows:*

$$x \mapsto f(g(x)).$$

For example, the function h is defined to be $h = f \circ g$ where $f(x) = x^2$ and $g(x) = x + 3$. This means $h(x) = f(g(x)) = (x + 3)^2$. To calculate $h(x)$ we first insert x as an input to g to obtain output $g(x)$, and then insert $g(x)$ as an input to f to obtain $h(x) = f(g(x))$.

Sometimes we can compose functions in either order. However, $g(f(x))$ may not be defined, even if $f(g(x))$ is defined. And when both expressions are defined, they need not be equal.

DEFINITION. *Given a function $f : X \to Y$ we say that f is* ***one-to-one*** *(or* ***injective****) if for any $x_1, x_2 \in X$*

$$f(x_1) = f(x_2) \implies x_1 = x_2.$$

Thinking of a function f as a machine which sends inputs to outputs, the corresponding "inverse function" sends each output back to the input from which it came. Not all functions have an associated inverse function. The formal definition is as follows:

DEFINITION. *A function f^{-1} : range$(f) \to$ domain(f) is the* ***inverse function*** *of f iff both of the following hold:*

$$f^{-1}(f(x)) = x \quad \forall x \in \text{domain}(f)$$
$$f(f^{-1}(x)) = x \quad \forall x \in \text{range}(f)$$

Said differently, the composition of a function and its inverse function (in either order) is the identity function.

We can define an inverse function for a given function f, provided that f is one-to-one. If f is not one-to-one, then it does not have an inverse.