Functions

Definition. *A* ***function*** *f consists of three sets (X, Y, R). X is called the* ***domain****. Y is called the* ***codomain****. $R \subset X \times Y$ consists of pairs of elements from X and Y, such that each $x \in X$ appears in exactly one pair $(x, y) \in R$ for some $y \in Y$. This formalism encodes the following intuition: you can think of a function as a machine which accepts inputs from X, and for each input it assigns precisely one output in Y.*

Let us introduce some notation. Instead of writing functions as triplets of sets, we usually denote them by $f : X \to Y$ where f is function with domain X and codomain Y. We often denote the elements of the domain by $x \in X$, and write $f(x) \in Y$ to denote the element to which x is assigned. This is referred to as the value of f at x. This is sometimes written

$$x \mapsto f(x).$$

We also refer to the element x which is being mapped as the argument of the function.

Definition. *Let $f : X \to Y$ be a function.*

(i) We say that f is ***injective*** *(or* ***one-to-one****) if $f(x) = f(y)$ implies $x = y$.*
(ii) We say that f is ***surjective*** *(or* ***onto****) if for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$.*
(iii) We say that f is ***bijective*** *if it is both injective and surjective.*

Definition. *Let $f : X \to Y$ be a function. We define the* ***image*** *of $S \subset X$ to be*

$$f(S) = \{f(x) \in Y : x \in S\}.$$

We say that $f(X) \subset Y$ is the ***range*** *of f.*

Note that a function is surjective iff its range is equal to its codomain.

If the range of a function g lies within the domain of a function f, then we can form the composition function $f \circ g$. This maps elements x in the domain of g as follows:

$$x \mapsto f(g(x)).$$

Definition. *Let $f : X \to Y$ be a function.*

(i) If there exists a function $g : Y \to X$ satisfying $(f \circ g)(y) = y \quad \forall y \in Y$, then we say that g is a ***right-inverse*** *for f.*
(ii) If there exists a function $h : Y \to X$ satisfying $(h \circ f)(x) = x \quad \forall x \in X$, then we say that h is a ***left-inverse*** *of f.*

Theorem. *Let $f : X \to Y$ be a function.*

(i) f is surjective iff it has a right-inverse.
(ii) f is injective iff it has a left-inverse.

Theorem. *If a function $f : X \to Y$ has a right-inverse $g : Y \to X$ and a left-inverse $h : Y \to X$, then $g \equiv h$.*

We refer to this unique function (when it exists) as the **inverse** of f, and it is denoted by $f^{-1} : Y \to X$. From the above it is clear that a function has an inverse iff it is bijective. To summarise, a function $f : X \to Y$ has an inverse $f^{-1} : Y \to X$ provided that

$$f^{-1}(f(x)) = x \;\; \forall x \in X$$
$$f(f^{-1}(y)) = y \;\; \forall y \in Y$$

Definition. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say that f is an* ***even function*** *if $f(x) = f(-x) \quad \forall x \in \mathbb{R}$. We say that f is an* ***odd function*** *if $f(x) = -f(-x) \quad \forall x \in \mathbb{R}$.*

Theorem. *Any function of the form $f : \mathbb{R} \to \mathbb{R}$ can be uniquely written as the sum of an even and an odd function.*

Definition. *Suppose $f : X \to \mathbb{R}$ is a function, where X is some subset of the real line. f is said to be* ***strictly increasing*** *if $f(a) > f(b)$ whenever $a > b$. f is said to be* ***strictly decreasing*** *if $f(a) < f(b)$ whenever $a > b$.*

Definition. *A* ***periodic*** *function is a function $f : \mathbb{R} \to \mathbb{R}$ such that there exists a $k > 0$ for which*

$$f(x + k) = f(x) \quad \forall x \in \mathbb{R}.$$

If there exists a least such k, then this value is called the ***period*** *of the function.*