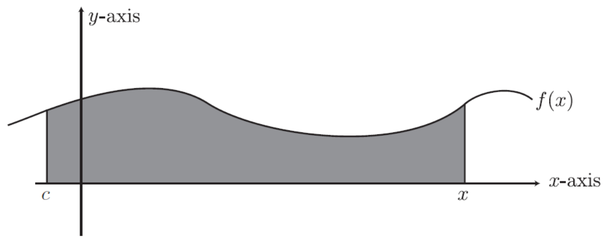
Idea of the Fundamental Theorem of Calculus: The diagram below shows the graph of a positive continuous function $f(x)$. Fix $c \in \mathbb{R}$ and let $F(x)$ denote the area enclosed beneath the graph, in between the two points c and x, and above the x-axis. This region is shaded below.



Suppose that $c < a < b$. The area under the graph of f from $x = a$ to $x = b$ is written $\int_a^b f(x)\,dx$. Directly from the definition of $F(x)$ we can conclude that

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

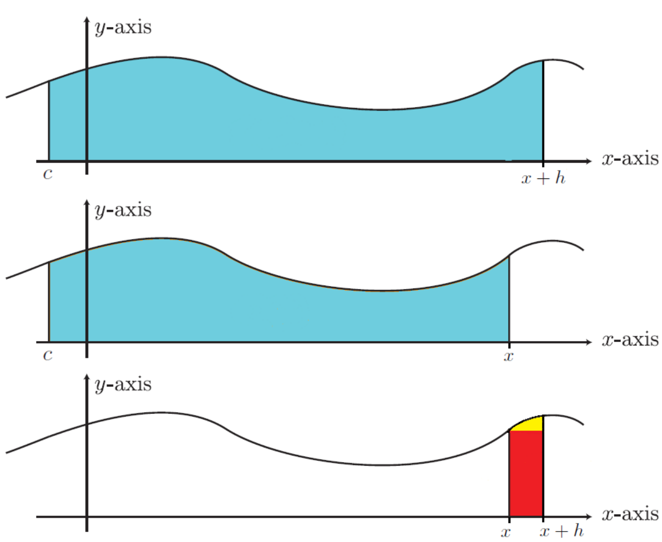
It turns out that the derivative of $F(x)$ is precisely $f(x)$, and this fact is so important that it is called the Fundamental Theorem of Calculus.

Proof of the Fundamental Theorem of Calculus: We need to investigate the limit

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}.$$

Note the following:

(i) $F(x+h)$ is equal to the shaded area in the first image below.
(ii) $F(x)$ is equal to the shaded area in the second image below.
(iii) $F(x+h) - F(x)$ is equal to the area of the red rectangle plus the area of the yellow region in the third image below.



The red rectangle has area $f(x) \cdot h$. Let $M = \max_{t\in[x,x+h]} f(t)$ and $m = \min_{t\in[x,x+h]} f(t)$. Then the area of the yellow region satisfies

$$\text{yellow area} \le (M - m)h.$$

Moreover, as $h \to 0$, we note that $M \to f(x)$ and $m \to f(x)$. Hence,

$$\lim_{h\to 0} \frac{\text{yellow area}}{h} = 0.$$

Therefore,

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\ &= \lim_{h\to 0} \frac{\text{red area} + \text{yellow area}}{h} \\ &= \lim_{h\to 0} \frac{f(x)\cdot h}{h} + \lim_{h\to 0} \frac{\text{yellow area}}{h} \\ &= f(x) + 0 \\ &= f(x) \end{aligned}$$

Our calculation above shows that the area function $F(x)$ is an antiderivative of $f(x)$, and we know by definition that

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Note any other antiderivative $G(x)$ of $f(x)$ differs from $F(x)$ by a constant, that is $G(x) = F(x) + c$ for some $c \in \mathbb{R}$. Now for the remarkable conclusion:

$$\int_a^b f(x)\,dx = F(b) - F(a) = \Big(G(b) - c\Big) - \Big(G(a) - c\Big) = G(b) - G(a).$$

Therefore, we can compute the area under the graph of f using *any* antiderivative of f! However, antidifferentiation is simply the inverse process of differentiation which, a priori, has nothing to do with computing areas! The Fundamental Theorem of Calculus gives an amazing connection between differentiation/antidifferentiation and the process of finding areas under curves. This is the great utility of our theorem, that given f, we can use any of its antiderivatives to compute the area under its graph. □

Formal Statement of the Fundamental Theorem of Calculus:

THEOREM. *Let f be a continuous real-valued function defined on some interval containing $c \in \mathbb{R}$. Define a new function F with the same domain*

$$F(x) = \int_c^x f(t)\,dt.$$

Then, $F(x)$ is differentiable and $F'(x) = f(x)$. Moreover, if F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This last expression is the signed area enclosed between the graph of f, the x-axis, and the lines $x = a$ and $x = b$. That is, it is the area enclosed above the x-axis minus the area enclosed below the x-axis.