Circle Geometry

Definition. *A **circle** is the set of all points in two-dimensional space that are a constant distance r from a fixed point. This distance is called the **radius** of the circle, and the fixed point is called the centre. A connected subset of a circle is called an **arc**. The angle associated with an arc that is formed at the centre of a circle is known as the angle **subtended** by the arc. A line segment joining two distinct points on a circle is called a **chord**. A line which intersects precisely two points on a circle is called a **secant**. A line which intersects precisely one point on a circle is called a **tangent** to the circle. The angle that a chord's endpoints make with any point on the circumference of the circle is called an angle **inscribed** by the chord. We will also refer to the angle that the endpoints of the chord make with the centre of the circle, as the angle **subtended** by the chord.*

Theorem. *If two circles are tangent at a point T, then T and the centres of the circles are collinear.*

Theorem. *For every circle the ratio of the circumference C to the diameter d is a constant value called pi and denoted π. That is, $C = \pi d$.*

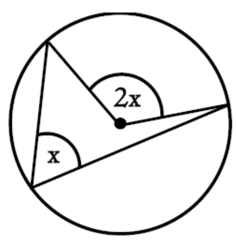
Theorem. *The area of a circle of radius r is πr^2.*

Theorem. *Two chords in a circle have equal length iff they subtend equal angles.*

Theorem. *Suppose a circle of radius r has an arc that subtends an angle $\theta < \pi$ (in radians). This arc defines a wedge shaped region, called a **sector**, and the hemispherical end of the sector is called a **segment**. We have the following formulas*

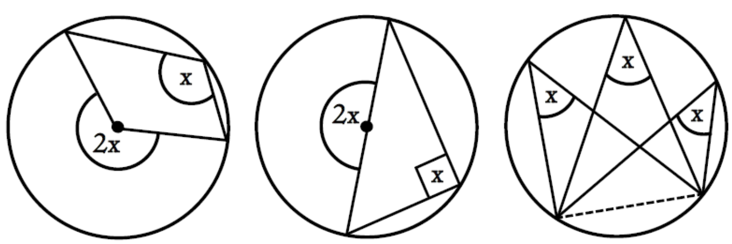
$$\begin{aligned} \textit{arc length} &= r\theta \\ \textit{chord length} &= 2r\sin\left(\frac{\theta}{2}\right) \\ \textit{area of sector} &= \frac{r^2\theta}{2} \\ \textit{area of segment} &= \frac{r^2}{2}\Big(\theta - \sin(\theta)\Big) \end{aligned}$$

Theorem. *The angle subtended by a chord is twice any angle inscribed by the chord (and on the same side of the chord).*



Corollary.

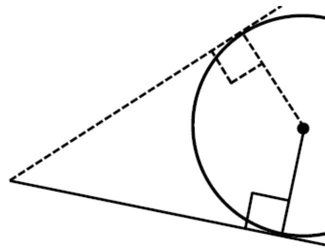
(i) If the angles above lie on opposite sides of the chord, then the reflex angle subtended by the chord is twice the angle inscribed by the chord.
(ii) Any angle inscribed by a diameter is a right angle.
(iii) Any two angles inscribed by the same chord (and on the same side of the chord) are equal.



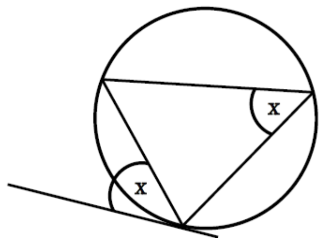
Definition. *A **cyclic quadrilateral** is a quadrilateral for which there exists a circle passing through all four of its vertices.*

Theorem. *Opposite angles in a cyclic quadrilateral are supplementary.*

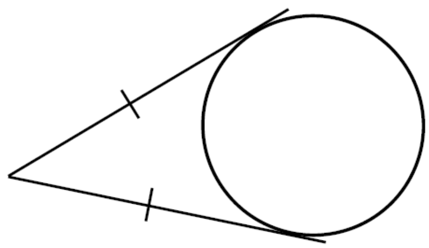
THEOREM. *A tangent to a circle is perpendicular to the radius through the point of tangency. Conversely, a line through a point on a circle which is perpendicular to the radius through that point is necessarily a tangent to the circle.*



THEOREM. *The angle formed by a tangent to a circle and a chord through the point of tangency is half the angle subtended by the chord.*



THEOREM. *Suppose two distinct lines are tangent to a circle and intersect outside the circle. The distances from the point of intersection of the lines to their points of intersection with the circle are equal.*



THEOREM. *Any three non-collinear points in the plane determine a triangle.*

(i) *There is a unique circle which passes through these points known as the* ***circumcircle*** *of the triangle.*
(ii) *There is a unique circle which is tangent to all three of its sides known as the* ***incircle*** *of the triangle.*

COROLLARY. *The lengths of the vertices of triangle* ΔABC *to the points of tangency of its incircle are given by* $s-a$, $s-b$, *and* $s-c$ *respectively, where* $s=\frac{1}{2}(a+b+c)$ *is the semiperimeter.*

THEOREM. *If* AB *and* CD *are two chords of a circle that intersect at a point* P *(which may be inside or outside the circle), then*

$$PA \cdot PB = PC \cdot PD.$$

Moreover, if AB *is a chord of a circle and* T *is a point on the circle such that* PT *is tangent to the circle, then*

$$PA \cdot PB = PT^2.$$