Complex Numbers

Definition. *The **complex numbers**, denoted by $\mathbb{C}$, may be defined as the set of elements $a + bi$ where $a, b \in \mathbb{R}$ and i satisfies $i^2 = -1$. Addition of complex numbers is defined as follows*

$$(a + bi) + (c + di) := (a + b) + (c + d)i,$$

and multiplication is defined by

$$(a + bi)(c + di) := (ac - bd) + (ad + bc)i.$$

Check that the definition of multiplication agrees with what you get if you expand the LHS above, and naively use the distributive law and the fact that $i^2 = -1$.

We define the following basic functions associated with a complex number $z = a + bi$.

(i) The **real part** of z is defined to be $\mathrm{Re}(z) = a$.
(ii) The **imaginary part** of z is defined to be $\mathrm{Im}(z) = b$.
(iii) The **magnitude** of z is defined to be $|z| = \sqrt{a^2 + b^2}$.
(iv) The **conjugate** of z is defined to be $\overline{z} = a - bi$.

Properties. *Let $z, w \in \mathbb{C}$.*

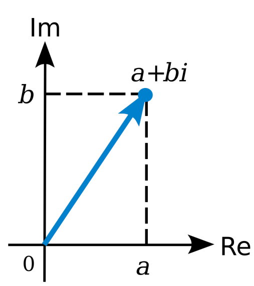
(i) $z = \mathrm{Re}(z) + \mathrm{Im}(z)i$.
(ii) $\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})$
(iii) $\mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z})$
(iv) $\overline{\overline{z}} = z$
(v) $\overline{z + w} = \overline{z} + \overline{w}$ *and* $\overline{z - w} = \overline{z} - \overline{w}$
(vi) $\overline{zw} = \overline{z}\overline{w}$ *and* $\overline{z/w} = \overline{z}/\overline{w}$
(vii) $|z| = \sqrt{z\overline{z}}$
(viii) $|z| \geq 0$ *with* $|z| = 0$ *iff* $z = 0$.
(ix) $|wz| = |w||z|$
(x) $|w/z| = |w|/|z|$
(xi) $w/z = w\overline{z}/(z\overline{z}) = \dfrac{1}{|z|^2}(w\overline{z})$

From the above properties it is clear that every non-zero complex number has a multiplicative inverse, that is, we can divide by any non-zero complex number.

Proposition (Triangle Inequality). *Let $w, z \in \mathbb{C}$.*

$$|w + z| \leq |w| + |z|.$$

We may think of a complex number as representing a point in the complex plane. The real part gives its horizontal component, and the imaginary part gives its vertical component.



We can also represent complex numbers using **polar form**. This associates to a complex number $z = a+bi$ an angle θ satisfying $\tan(\theta) = b/a$ and a radius r satisfying $r = \sqrt{a^2 + b^2} = |z|$, such that

$$z = a + bi = r(\cos(\theta) + i\sin(\theta)).$$

If you think of the complex number z being connected to the origin by a line segment, then θ is the angle this line segment makes with the positive direction of the x-axis, and r is the magnitude of z. Note that sometimes $\cos(\theta) + i\sin(\theta)$ is abbreviated as $\text{cis}(\theta)$.

Analogous to the real setting, their exists a complex exponential function $e : \mathbb{C} \to \mathbb{C}$ defined by

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

The following celebrated formula relates the **complex exponential function** to the polar representation.

Theorem (Euler's formula).

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Corollary.

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$
$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

The above formula implies that the complex number $z = r\,\text{cis}(\theta)$ may also be written as $z = re^{i\theta}$. This polar representation does not assist with addition or subtraction of complex numbers, but it does provide a much more concrete picture of what multiplication and division of complex numbers means. Suppose $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Then,

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}.$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.$$

Also,

$$\overline{z} = \overline{re^{i\theta}} = re^{-i\theta}$$

Theorem (De Moivre). *Suppose $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then*

$$\Big(\cos(x) + i\sin(x)\Big)^n = \cos(nx) + i\sin(nx)$$

which implies

$$(re^{i\theta})^n = r^n\Big(\cos(x) + i\sin(x)\Big)^n = r^n\Big(\cos(nx) + i\sin(nx)\Big) = r^n e^{in\theta}.$$

Corollary. *The n^{th} roots of a complex number $w = re^{i\theta}$ are given by*

$$r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

where $0 \le k \le n-1$. These are the roots of the polynomial $z^n - w$.

We define the **$\mathbf{n^{th}}$ roots of unity** to be the n distinct roots of the polynomial $z^n - 1$. From the above corollary these are equally spaced around the unit circle in the complex plane. When dealing with roots of unity we can often make use of the following factorization:

$$z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \cdots + z^2 + z + 1).$$

The complex numbers were born out of a desire to solve equations such as $x^2 + 1 = 0$. It would seem reasonable that because certain polynomials with real coefficients necessitated complex numbers to solve, that there would exist polynomials with complex coefficients which require an even larger number system to solve. Remarkably, this turns out not to be the case, once we include i, we can solve all (non-constant) polynomials for no extra effort.

Theorem (Fundamental Theorem of Algebra). *For every non-constant polynomial with real or complex coefficients $p(z)$, there exists $z \in \mathbb{C}$ such that $p(z) = 0$.*

Corollary. *Every polynomial with real or complex coefficients $p(z)$ of degree $n \in \mathbb{N}$ can be written as*

$$p(z) = \lambda(z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_n)$$

where $\lambda, \alpha_1, \cdots, \alpha_n \in \mathbb{C}$ (not necessarily distinct).