Exponentials & Logarithms

Definition. *Let $a \in (0,1) \cup (1,\infty)$. A function of the form $f : \mathbb{R} \to \mathbb{R}$ $f(x) = a^x$ is called an* ***exponential function****. Intuitively, it is used to model compounding growth (or decay).*

Definition. *Let $a \in (0,1) \cup (1,\infty)$. A function of the form $g : (0,\infty) \to \mathbb{R}$ $g(x) = \log_a(x)$ is called a* ***logarithm function****, where $y = \log_a(x)$ is simply alternative notation for $a^y = x$. Intuitively, you should think of the logarithm as telling you "the number of powers of "a" which multiply to give x".*

(i) Suppose $a \in (1,\infty)$. Then the associated exponential and logarithm functions are both increasing. That is, $x_1 > x_2$ implies $a^{x_1} > a^{x_2}$ and $\log_a(x_1) > \log_a(x_2)$.
(ii) Suppose $a \in (0,1)$. Then the associated exponential and logarithm functions are both decreasing. That is, $x_1 > x_2$ implies $a^{x_1} < a^{x_2}$ and $\log_a(x_1) < \log_a(x_2)$.

Also note that similar statements hold even if the inequalities aren't strict. As a consequence both functions are one-to-one.

$$a^{x_1} = a^{x_2} \iff x_1 = x_2$$
$$\log_a(x_1) = \log_a(x_2) \iff x_1 = x_2$$

In fact, the exponential and logarithm functions are inverses to one another. That is, by definition,

$$\log_a(a^x) = x \quad \forall x \in \mathbb{R}$$
$$a^{\log_a(x)} = x \quad \forall x \in (0,\infty)$$

Essentially, a logarithm is an algebraic device for turning problems involving multiplication into problems involving addition. The laws of logarithms below follow more or less directly from the index laws.

(i) $\log_a(x_1 x_2) = \log_a(x_1) + \log_a(x_2)$
(ii) $\log_a(x_1/x_2) = \log_a(x_1) - \log_a(x_2)$
(iii) $\log_a(x^k) = k\log_a(x)$
(iv) $\log_{a^k}(x^k) = \log_a(x)$
(v) $\log_a(1) = 0$
(vi) $\log_a(a) = 1$

Logarithms transform via the change of base formula

$$\log_a(x) = \frac{1}{\log_b(a)}\log_b(x).$$

We can also think of the logarithms as being chained together

$$\log_b(a)\log_a(x) = \log_b(x).$$

In particular, choosing $x = b$ we obtain

$$\log_b(a) = \frac{1}{\log_a(b)}.$$

In addition, choosing $b = 1/a$ we obtain

$$\log_a(x) = -\log_{\frac{1}{a}}(x).$$

Exponentials transform via the change of base formula

$$a^x = b^{\log_b(a)x}.$$

The above formulas imply that the graph of $y = \log_a(x)$ is obtained from the graph of $y = \log_b(x)$ via a dilation by a factor of $1/\log_b(a)$ from the x-axis. In addition, the graph of $y = a^x$ is obtained from the graph of $y = b^x$ via a dilation by a factor of $1/\log_b(a)$ from the y-axis. Hence, up to these dilation factors exponentials and logarithms with respect to different bases are the same. For this reason we often make a particularly judicious choice of base and use the unique positive real number defined as

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.$$

This is known as the mathematical constant e. It plays a fundamental role throughout mathematics. The primary reason we work with this number as a base is that it yields greatly simplified formulas in calculus.