Limits & Continuity

Limit of a sequence.

Definition. *Formally, a* ***sequence*** $a_0, a_1, a_2, ...$ *of real numbers is a mapping from* $\mathbb{N} \to \mathbb{R}$. *That is, a sequence of real numbers is an ordered collection of real numbers. We will use the notation* $(a_n)_{n\in\mathbb{N}}$ *to denote a sequence.*

Most interesting questions to do with sequences concern their long term behaviour.

Definition. *A sequence* $(a_n)_{n\in\mathbb{N}}$ *has a* ***limit*** $L \in \mathbb{R}$ *if* $\forall\varepsilon > 0,\ \exists N \in \mathbb{N}$ *such that*

$$|a_n - L| < \varepsilon$$

whenever $n > N$. *In this situation we say that* $(a_n)_{n\in\mathbb{N}}$ ***converges*** *to* L*, and write*

$$\lim_{n\to\infty} a_n = L.$$

If a sequence does not converge to any limit, then we say that it is ***divergent****.*

Definition. *Given a sequence* $(a_n)_{n\in\mathbb{N}}$*, if* $\forall M \in \mathbb{N},\ \exists N \in \mathbb{N}$ *such that*

$$a_n > M$$

whenever $n > N$*, then we say that* $(a_n)_{n\in\mathbb{N}}$ *diverges to* ∞ *and write*

$$\lim_{n\to\infty} a_n = \infty.$$

Limit of a function.

Here we introduce the concept of the limiting value of a function as its argument approaches a fixed point. It is similar to the limit of a sequence.

Definition. *Let* $f : (a,b) \to \mathbb{R}$ *be a function and fix* $x_0 \in (a,b)$. *We say that the* ***limit*** *of* $f(x)$ *as* x *approaches* x_0 *is* $L \in \mathbb{R}$ *and write*

$$\lim_{x\to x_0} f(x) = L$$

if $\forall\varepsilon > 0,\ \exists\delta > 0$ *such that*

$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - x_0| < \delta.$$

Intuitively, the defining condition means we can make $f(x)$ as close to L as we like, provided we can shrink the distance from x to x_0 to be sufficiently small.

Remark. *Note the following carefully:*

- *(i) For any given point* x_0 *in its domain a function* f *may not have a limit. Give an example.*
- *(ii) Limits, when they exist, are unique. Prove this.*
- *(iii) The quantity* $\lim_{x\to x_0} f(x)$ *only depends on the values of* f *"close to"* x_0*, but not the value at* x_0*.*
- *(iv) Even when a function* f *has a limit at* x_0*, this may not equal* $f(x_0)$*. Give an example.*
- *(v) When this last condition is satisfied we say that* f *is continuous (see below). The concept of a continuous function is fundamental in modern mathematics.*

Theorem. *Let* $f : (a,b) \to \mathbb{R}$ *be a function and fix* $x_0 \in (a,b)$*. The following are equivalent:*

- *(i)* $\lim_{x\to x_0} f(x) = L$.
- *(ii) For every sequence* $(c_n)_{n\in\mathbb{N}}$ *taking values in* (a,b) *with* $\lim_{n\to\infty} c_n = x_0$*, we have that* $\lim_{n\to\infty} f(c_n) = L$.

Theorem (Limit Laws). *Suppose* $f, g : (a,b) \to \mathbb{R}$ *are functions,* $x_0 \in (a,b)$ *and*

$$\lim_{x\to x_0} f(x) = L \quad \text{and} \quad \lim_{x\to x_0} g(x) = M$$

where $L, M \in \mathbb{R}$*. Then,*

$$\lim_{x\to x_0} \Big(f(x) + g(x)\Big) = L + M$$

$$\lim_{x\to x_0} f(x)g(x) = LM$$

$$\lim_{x\to x_0} kf(x) = kL \quad \forall k \in \mathbb{R}$$

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{L}{M} \quad \text{for } M \neq 0.$$

Theorem (Squeeze Theorem). *Suppose $f(x) \leq g(x) \leq h(x)$ for all x near x_0 and $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = L$. Then*

$$\lim_{x\to x_0} g(x) = L.$$

Continuity.

Definition. *Let $f : (a,b) \to \mathbb{R}$ be a function and fix $x_0 \in (a,b)$. We say that f is* ***continuous*** *at x_0 if $\lim_{x\to x_0} f(x) = f(x_0)$. If a function is continuous at all points x_0 in its domain, then we simply say that is a continuous function.*

Intuitively, if a function is continuous, you can determine its value at a point even if a priori you only know its values at nearby points. Almost all of the functions that you encounter in secondary school are continuous i.e. polynomials, sine/cosine, exponentials etc.

Corollary. *Let $f : (a,b) \to \mathbb{R}$ be a function and fix $x_0 \in (a,b)$. The following are equivalent:*

(i) f is continuous at x_0.
(ii) For every sequence $(c_n)_{n\in\mathbb{N}}$ taking values in (a,b) with $\lim_{n\to\infty} c_n = x_0$, we have that $\lim_{n\to\infty} f(c_n) = f(x_0)$.

That is, continuous functions map converging sequences $(c_n)_{n\in\mathbb{N}}$ to converging sequences $(f(c_n))_{n\in\mathbb{N}}$.

L'Hôpital's Rule.

The following theorem gives a very efficient technique for evaluating certain limits.

Theorem (L'Hôpital's Rule). *Suppose f and g are differentiable (except possibly at x_0) real-valued functions satisfying*

$$\lim_{x\to x_0} f(x) = 0 \quad \textit{and} \quad \lim_{x\to x_0} g(x) = 0$$

or,

$$\lim_{x\to x_0} f(x) = \pm\infty \quad \textit{and} \quad \lim_{x\to x_0} g(x) = \pm\infty.$$

Then,

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right exists (and $g'(x) \neq 0$ when x is close to x_0).