Proofs

Mathematical **statements** are often denoted by capital letters i.e. P, Q, and may be either true or false. The **negation** of a statement P is written "not P", and is the opposite of P. We can also form statements that involve other statements. For example, a **conditional statement** is a statement that has the form:

$$\text{"If } P, \text{ then } Q\text{"}$$

where P, Q are two statements. This is also written

$$P \implies Q$$

and interpreted as P **implies** Q. In order to establish the truth of a conditional statement, we must show that if P is true, then it follows that Q must be true.

Given a conditional statement: $P \implies Q$, then the **converse** is the conditional statement:

$$Q \implies P.$$

Note carefully that the truth of the original statement does not imply the truth of the converse. Two statements P, Q are said to be **equivalent** if $P \implies Q$ and $Q \implies P$. This is also written as

$$P \iff Q$$

and read "P is true if and only if (iff) Q is true."

Given a conditional statement: $P \implies Q$, then the **contrapositive** statement is the following conditional statement:

$$\text{not } Q \implies \text{not } P.$$

Note that the original statement and the contrapositive are logically equivalent. Hence, proving one establishes the truth of both. Sometimes we prove the contrapositive instead of the original statement because it can be easier.

A **proof by contradiction** follows the format:

(1) Assume that the statement we want to prove is false.
(2) Show that this assumption leads to a contradiction.
(3) Conclude that our initial assumption must have been false, and thus the statement we want to prove is true.

A **proof by induction** follows the format:

(1) Define a sequence of statements, denoted $P(n)$, one for each $n \in \mathbb{N}$.
(2) Base case: show that $P(1)$ is true.
(3) Inductive step: assume that $P(k)$ is true, and under this assumption show that $P(k+1)$ must be true.
(4) Conclude that $P(n)$ is true for all $n \in \mathbb{N}$.