Vector Calculus

Basics.

Definition. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a real-valued function. We define the* ***gradient*** *of f to be*

$$\nabla f := \Big(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\Big).$$

Definition. *Let $\mathbf{F} = (F_1, F_2, F_3) : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field.*

(i) We define the ***divergence*** *of $\mathbf{F}$ to be*

$$\operatorname{div}(\mathbf{F}) := \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

(ii) We define the ***curl*** *of $\mathbf{F}$ to be*

$$\operatorname{curl}(\mathbf{F}) := \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\mathbf{i} + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\mathbf{j} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\mathbf{k}.$$

Remark. *Intuitively, if you imagine a vector field as representing the flow of some quantity, then the divergence of the vector field at a point represents the net amount of that quantity flowing out from the point. When the divergence is positive, then there is a net outward flow from the point (i.e. the point is a source). When the divergence is negative, then there is a net inward flow at the point (i.e. the point is a sink). Under this interpretation the curl of the vector field at a point represents the axis about which the quantity rotates (according to the right-hand rule) as it flows. The extent of the rotation is determined by the magnitude of the curl of the vector field.*

Definition. *If a vector field $\mathbf{F}$ satisfies $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$, then it is said to be* ***irrotational****. If a vector field $\mathbf{F}$ satisfies $\operatorname{div}(\mathbf{F}) = 0$, then it is said to be* ***incompressible****.*

Definition. *Assume f is a real-valued function, and $\mathbf{F} = (F_1, F_2, F_3)$ is a vector field. Then the* ***Laplace operator*** *Δ is defined as follows*

$$\Delta f := \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2},$$

$$\Delta \mathbf{F} := \Big(\Delta F_1, \Delta F_2, \Delta F_3\Big).$$

A real-valued function f is said to be ***harmonic*** *if $\Delta f = 0$.*

Theorem. *Suppose $\mathbf{F}, \mathbf{G}$ are vector fields and f, g are scalar functions. Suppose all partial derivatives exist and are continuous. Then,*

(i) $\nabla(fg) = g\nabla f + f\nabla g$

(ii) $\operatorname{div}(f\mathbf{F}) = f\operatorname{div}(\mathbf{F}) + \mathbf{F}\cdot(\nabla f)$

(iii) $\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \operatorname{curl}(\mathbf{F})\cdot\mathbf{G} - \mathbf{F}\cdot\operatorname{curl}(\mathbf{G})$

(iv) $\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}(\mathbf{F}) + (\nabla f)\times\mathbf{F}$

(v) $\operatorname{curl}(\mathbf{F}\times\mathbf{G}) = \operatorname{div}(\mathbf{G})\mathbf{F} - \operatorname{div}(\mathbf{F})\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G}$

(vi) $\nabla(\mathbf{F}\cdot\mathbf{G}) = \mathbf{F}\times\operatorname{curl}(\mathbf{G}) + \mathbf{G}\times\operatorname{curl}(\mathbf{F}) + (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F}$

(vii) $\operatorname{curl}(\operatorname{curl}(\mathbf{F})) = \nabla(\operatorname{div}(\mathbf{F})) - \Delta\mathbf{F}$

(viii) $\Delta(fg) = f(\Delta g) + 2(\nabla f)\cdot(\nabla g) + g(\Delta f)$

(ix) $\operatorname{div}((\nabla f)\times(\nabla g)) = 0$

(x) $\operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0$

(xi) $\operatorname{curl}(\nabla f) = \mathbf{0}$

Definition. *Suppose a curve is described by the parametric equations* $\boldsymbol{\gamma}(t) = (x(t), y(t), z(t))$ $a \leq t \leq b$ *where* $x'(t)$, $y'(t)$ *and* $z'(t)$ *are continuous on* $[a,b]$. *Then the **length** of* $\boldsymbol{\gamma}$ *is given by*

$$L = \int_{\boldsymbol{\gamma}} ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

Definition. *The surface area of the solid obtained by rotating about the* x*-axis the region under the curve* $y = f(x)$ *from* $x = a$ *to* $x = b$ *can be computed via the formula*

$$A = \int_a^b 2\pi f(x) \sqrt{1 + \big(f'(x)\big)^2}\, dx.$$

A similar formula holds for rotations about the y*-axis.*

Definition. *The volume of the solid obtained by rotating about the* x*-axis the region under the curve* $y = f(x)$ *from* $x = a$ *to* $x = b$ *can be found by integrating the cross-sectional area as a function of* x

$$V = \int_a^b \pi \big(f(x)\big)^2\, dx$$

This volume can also be computed via the formula

$$V = \int_a^b 2\pi x f(x)\, dx.$$

Similar formulas hold for rotations about the y*-axis.*

Definition.

(i) *We define the **line integral** of a function* f *along a smooth curve* $\boldsymbol{\gamma}(t) = (x(t), y(t), z(t))$ *as*

$$\int_{\boldsymbol{\gamma}} f(x,y,z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_a^b f(\boldsymbol{\gamma}(t))|\boldsymbol{\gamma}'(t)|\, dt.$$

We can also compute line integrals along $\boldsymbol{\gamma}$ *with respect to coordinate directions as follows*

$$\int_{\boldsymbol{\gamma}} f(x,y,z)\, dx = \int_a^b f(x(t), y(t), z(t)) x'(t)\, dt.$$

(ii) *We define the line integral of a vector field* $\mathbf{F} = (F_1, F_2, F_3)$ *along a smooth curve* $\boldsymbol{\gamma}$ *as*

$$\int_{\boldsymbol{\gamma}} \mathbf{F} \cdot \mathbf{ds} = \int_{\boldsymbol{\gamma}} \mathbf{F}(\boldsymbol{\gamma}(t)) \cdot \frac{\boldsymbol{\gamma}'(t)}{|\boldsymbol{\gamma}'(t)|}\, ds = \int_a^b \mathbf{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\, dt = \int_{\boldsymbol{\gamma}} F_1(\boldsymbol{\gamma}(t)) dx + \int_{\boldsymbol{\gamma}} F_2(\boldsymbol{\gamma}(t)) dy + \int_{\boldsymbol{\gamma}} F_3(\boldsymbol{\gamma}(t)) dz.$$

Note that these integrals will depend on the path $\boldsymbol{\gamma}$, but they will not depend on the parametrisation of that path, provided that the orientation is preserved.

Theorem (Fundamental Theorem for Line Integrals). *Suppose* $\boldsymbol{\gamma}$ *is a piecewise smooth curve defined on* $[a,b]$, *and* f *is a real-valued function with continuous derivatives. Then,*

$$\int_{\boldsymbol{\gamma}} \nabla f \cdot \mathbf{ds} = f(\boldsymbol{\gamma}(b)) - f(\boldsymbol{\gamma}(a)).$$

Definition.

(i) *The surface integral of a function* $f : \mathbb{R}^3 \to \mathbb{R}$ *over a parametric surface* S *described by* $\varphi(u,v) : D \to \mathbb{R}^3$ *is*

$$\iint_S f(x,y,z)\, dS = \iint_D f(\varphi(u,v)) |\varphi_u \times \varphi_v|\, dA$$

(ii) *Suppose* $\mathbf{F}$ *is a continuous vector field defined on an oriented surface* S *with parametric description* $\varphi(u,v) : D \to \mathbb{R}^3$ *and unit normal vector*

$$\mathbf{n} = \frac{\varphi_u \times \varphi_v}{|\varphi_u \times \varphi_v|}.$$

Then the surface integral of $\mathbf{F}$ *over* S *is*

$$\iint_S \boldsymbol{F} \cdot \mathbf{dS} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D \mathbf{F} \cdot (\varphi_u \times \varphi_v)\, dA.$$

This integral is also called the flux of $\mathbf{F}$ *across* S.

Note that these integrals will not depend on the particular parametrisation of the surface that is used.

Conservative Vector Fields.

DEFINITION. *A vector field* $\mathbf{F}$ *is called* ***conservative*** *if* $\mathbf{F} = \nabla f$ *for some real-valued function* f.

THEOREM. *Suppose* $\mathbf{F}$ *is a vector field in an open connected region* $D \subset \mathbb{R}^n$. *The following are equivalent:*

(i) $\mathbf{F}$ *is conservative.*

(ii) For every piecewise smooth closed curve $\boldsymbol{\gamma}$ *in* D *we have*

$$\int_{\boldsymbol{\gamma}} \mathbf{F} \cdot \mathbf{ds} = 0.$$

(iii) For any two points $p, q \in D$, *and for any two piecewise smooth curves* $\boldsymbol{\gamma_1}$ *and* $\boldsymbol{\gamma_2}$ *that start at* p *and end at* q, *we have*

$$\int_{\boldsymbol{\gamma_1}} \mathbf{F} \cdot \mathbf{ds} = \int_{\boldsymbol{\gamma_2}} \mathbf{F} \cdot \mathbf{ds}.$$

THEOREM. *Suppose* $\mathbf{F}$ *is a vector field defined on a simply connected region in* $\mathbb{R}^3$ *whose components have continuous partial derivatives. Then,* $\mathbf{F}$ *is conservative iff* $\operatorname{curl} \mathbf{F} = 0$.

DEFINITION. *A vector field* $\mathbf{F}$ *is said to be* ***radial*** *if*

$$\mathbf{F}(x, y, z) = f(r)\hat{\mathbf{r}}(x, y, z)$$

where $\mathbf{r}(x, y, z) = (x, y, z)$, $\hat{\mathbf{r}} = \dfrac{\mathbf{r}}{|\mathbf{r}|}$ *and* $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.

PROPOSITION. *All radial vector fields are conservative.*

Proof. Suppose $\mathbf{F}(x, y, z) = f(r)\hat{\mathbf{r}}(x, y, z)$ is a radial vector field. Define $\varphi(x, y, z) = \displaystyle\int_0^r f(t)\,dt$. Then,

$$\begin{aligned}
\nabla\varphi &= f(r)\nabla r \\
&= f(r)\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) \\
&= f(r)\hat{\mathbf{r}}(x, y, z) \\
&= \mathbf{F}
\end{aligned}$$

□

REMARK. *Many physical phenomena (gravitation, sound intensity, electrostatics, etc.) are governed by an inverse square law. The commonality in these phenomena is that there exists a point source that is propagating something uniformly outwards in three dimensional space. The propagated quantity is diluted as the distance* r *from the source increases, since it is spread over an expanding spherical shell of area* $4\pi r^2$. *These phenomena can each be described by a vector field of the form*

$$\mathbf{F} = \frac{k}{r^2}\hat{\mathbf{r}}$$

where $k \in \mathbb{R}\setminus\{0\}$. *In general, suppose we have a point source that is propagating something uniformly outwards in* n*-dimensional space. That is, the propagated quantity doesn't accumulate, but expands outward as a sphere of increasing radius. The surface area of the* $(n-1)$*-sphere of radius* r *is given by* r^{n-1} *multiplied by some constant. Hence, we would expect such a phenomena to be governed by a power law of the form* $1/r^{n-1}$. *The following theorem shows that such vector fields have divergence zero, as we expect from our surface area heuristic.*

THEOREM. *Suppose* $\mathbf{r}(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n)$, *so that* $r = |\mathbf{r}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. *Consider the vector field*

$$\mathbf{F} = \frac{k}{r^p}\hat{\mathbf{r}}$$

where $p \in \mathbb{Z}$ *and* $k \in \mathbb{R}\setminus\{0\}$. *Then,*

$$\operatorname{div} \mathbf{F} = 0$$

iff $p = n - 1$.

Change of Variables.

Theorem (Change of Variables I). *Let $[a,b] \subset \mathbb{R}$ be an interval and $\varphi : [a,b] \to \mathbb{R}$ an injective continuously differentiable function for which $\varphi'(u) \neq 0$ for all $u \in [a,b]$. Suppose $f : \varphi([a,b]) \to \mathbb{R}$ is continuous. Then,*

$$\int_{\varphi([a,b])} f(v)\,dv = \int_{[a,b]} f(\varphi(u))|\varphi'(u)|\,du.$$

Theorem (Change of Variables II). *Let $U \subset \mathbb{R}^n$ be an open set, and $\varphi : U \to \mathbb{R}^n$ an injective continuously differentiable function for which $\det(D\varphi)(u) \neq 0$ for all $u \in U$. Suppose f is a continuous real-valued function with compact support whose support lies in $\varphi(U)$. Then,*

$$\int_{\varphi(U)} f(v)\,dv = \int_U f(\varphi(u))|\det(D\varphi)(u)|\,du.$$

Definition. *Cylindrical coordinates:* $0 \leq r \leq \infty,\ 0 \leq \theta \leq 2\pi,\ -\infty \leq z \leq \infty$

(i) $x = r\cos(\theta)$
(ii) $y = r\sin(\theta)$
(iii) $z = z$
(iv) $r^2 = x^2 + y^2$
(v) $\theta = \tan^{-1}(\frac{y}{x}) + k\pi$, *for some* $k = -1, 0, 1$.
(vi) $dx\,dy\,dz \mapsto r dr\,d\theta\,dz$

Definition. *Spherical coordinates:* $0 \leq \rho \leq \infty,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \varphi \leq \pi$

(i) $x = \rho\sin(\varphi)\cos(\theta)$
(ii) $y = \rho\sin(\varphi)\sin(\theta)$
(iii) $z = \rho\cos(\varphi)$
(iv) $\rho^2 = x^2 + y^2 + z^2$
(v) $\theta = \tan^{-1}(\frac{y}{x}) + k\pi$, *for some* $k = -1, 0, 1$.
(vi) $\varphi = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) + m\pi$, *for some* $m = -1, 0, 1$.
(vii) $dx\,dy\,dz \mapsto \rho^2\sin(\varphi)d\rho\,d\theta\,d\varphi$

Main Integration Theorems.

Theorem (Green's Theorem). *Let $\boldsymbol{\gamma}$ be a positively oriented (that is, counter-clockwise), piecewise-smooth, simple closed curve in the plane and let D be the region bounded by $\boldsymbol{\gamma}$. If $\mathbf{F} = (F_1, F_2)$ is a vector field with continuous partial derivatives on an open region that contains D, then*

$$\int_{\boldsymbol{\gamma}} \mathbf{F}\cdot\mathbf{ds} = \int_{\boldsymbol{\gamma}} F_1\,dx + F_2\,dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA.$$

An alternative formulation of the theorem is as follows:

$$\int_{\boldsymbol{\gamma}} \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \operatorname{div}\mathbf{F}\,dA$$

where $\mathbf{n}$ is the outward unit normal vector to $\boldsymbol{\gamma}$. That is, if $\boldsymbol{\gamma}(t) = (x(t), y(t))$, then

$$\mathbf{n}(t) = \frac{1}{|\boldsymbol{\gamma}'(t)|}\Big(y'(t), -x'(t)\Big).$$

Corollary. *Let $\boldsymbol{\gamma}$ be a positively oriented (that is, counter-clockwise), piecewise-smooth, simple closed curve in the plane and let D be the region bounded by $\boldsymbol{\gamma}$. The area of D is*

$$\frac{1}{2}\int_D (x\,dy - y\,dx).$$

Theorem (Stoke's Theorem). *Let S be an oriented piecewise smooth surface that is bounded by a simple, closed piecewise-smooth boundary curve ∂S (with the induced orientation). Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ containing S. Then,*

$$\int_{\partial S} \mathbf{F}\cdot\mathbf{ds} = \iint_S \operatorname{curl}\mathbf{F}\cdot\,\mathbf{dS}.$$

Theorem (Divergence Theorem). *Let Ω be a compact region with closed piecewise smooth boundary surface S, whose orientation is given by the outward pointing normal vector. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains Ω. Then,*

$$\int_S \mathbf{F}\cdot\,\mathbf{dS} = \iiint_\Omega \operatorname{div}\mathbf{F}\,dV.$$