Sequences & Series

Definition. *A **sequence** is an ordered collection of real numbers. A sequence does not need to have any particular pattern. We denote the first term of a sequence by t_1, the second term of a sequence by t_2, and in general, the n^{th} term by t_n. We denote the sum of the first n terms of a sequence by*

$$S_n = \sum_{k=1}^{n} t_k = t_1 + t_2 + t_3 + \cdots + t_n$$

*In general, we refer to the sum of a sequence as a **series**, and S_n as the n^{th} partial sum of the series.*

Examples The following are examples of sequences of real numbers.

(i) 4, 7, 87, -9000, π, 0, -5, -1,...
(ii) 1, 2, 3, -666, 4, 5, 6,...
(iii) 18, 100, -1, 23, 315, 1000000,...

Definition. *An **arithmetic sequence** is a sequence for which the next term can be obtained by adding a fixed constant to the previous term. A finite sum of an arithmetic sequence is called an **arithmetic series.***

Examples The following are examples of arithmetic sequences of real numbers.

(i) 1, 3, 5, 7, 9,...
(ii) 4, 10, 16, 22,...
(iii) 3, 1, -1, -3, -5,...

When dealing with arithmetic sequences we usually denote the common difference by d, so $t_{n+1} - t_n = d$, and the first term by a, so $t_1 = a$. All arithmetic sequences therefore have the following form:

$$a,\ a+d,\ a+2d,\ a+3d, \ldots$$

In general, the n^{th} term of an arithmetic sequence is given by

$$t_n = a + (n-1)d.$$

Proposition. *We can compute the sum of the first n terms of an arithmetic sequence by the following formula:*

$$S_n = \frac{n}{2}\Big(2a + (n-1)d\Big)$$

Definition. *A **geometric sequence** is a sequence for which the next term can be obtained by multiplying the previous term by a fixed non-zero constant. The sum of a geometric sequence is called a **geometric series.***

Examples The following are examples of geometric sequences of real numbers.

(i) 2, 4, 8, 16, 32,...
(ii) 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$,...
(iii) -5, 25, -125, 625, -3125,...

When dealing with geometric sequences we usually denote the common ratio by r, so $\dfrac{t_{n+1}}{t_n} = r$, and the first term by a, so $t_1 = a$. All geometric sequences therefore have the following form:

$$a,\ ar,\ ar^2,\ ar^3, \ldots$$

In general, the n^{th} term of a geometric sequence is given by

$$t_n = ar^{n-1}.$$

Proposition. *We can compute the sum of the first n terms of a geometric sequence by the following formula:*

$$S_n = \frac{a(1-r^n)}{1-r}$$

Remark. *Suppose we have a geometric series with common ratio satisfying $|r| < 1$, then the partial sums converge, and we can compute the infinite sum as follows*

$$S_\infty = \sum_{k=1}^{\infty} t_k = \frac{a}{1-r}.$$