Euler's Formula

You should be familiar with the **exponential function**

$$e^x : \mathbb{R} \to \mathbb{R}.$$

This has the following properties:

(1) $e^{x+y} = e^x e^y$ where $x, y \in \mathbb{R}$

(2) $\dfrac{d}{dx}\left(e^{kx}\right) = ke^{kx}$ where $k \in \mathbb{R}$, $x \in \mathbb{R}$.

Furthermore, recall that the set of **complex numbers** is defined as

$$\mathbb{C} := \left\{a + bi : a, b \in \mathbb{R}, i^2 = -1\right\}.$$

Multiplying two complex numbers yields another complex number as follows

$$\begin{aligned}(a+bi)(c+di) &= ac + bci + adi + bdi^2 \\ &= (ac - bd) + (bc + ad)i.\end{aligned}$$

It turns out that we can also exponentiate complex numbers, and thus extend the exponential function, mentioned above, to take values in the complex plane. This can be achieved in such a way that the previous two properties are upheld. Namely, there exists a **complex exponential function**

$$e^z : \mathbb{C} \to \mathbb{C}$$

satisfying:

(1) $e^{z+w} = e^z e^w$ where $z, w \in \mathbb{C}$.

(2) $\dfrac{d}{dx}\left(e^{ikx}\right) = ike^{ikx}$ where $k \in \mathbb{R}$, $x \in \mathbb{R}$.

Note that a particular consequence of (1) above is that $e^{i(x+y)} = e^{ix}e^{iy}$ where $x, y \in \mathbb{R}$.

The following exercise manifests the beautiful link between the complex exponential function and the usual trigonometric functions.

Exercise. *Define $f : \mathbb{R} \to \mathbb{C}$ by $f(x) := \cos(x) + i\sin(x)$. Show that $f(x)e^{-ix}$ is a constant by showing*

$$\frac{d}{dx}\left(f(x)e^{-ix}\right) = 0.$$

Evaluate $f(x)e^{-ix}$ at a point to conclude that

$$e^{ix} = \cos(x) + i\sin(x).$$

This celebrated result is known as ***Euler's formula****.*

This exercise shows that e^{ix} parametrizes the unit circle in the complex plane (when $x \in \mathbb{R}$). This is in stark contrast to the usual real exponential function which is unbounded on the real line.

Moreover, since any non-zero complex number is a point on the unit circle after some scaling, we can write any non-zero complex number as re^{ix} for some $r > 0$ and some $x \in \mathbb{R}$. This is called its **polar representation**. This representation allows us to develop an intuitive picture of what is occurring under complex multiplication. Suppose we multiply a complex number $r_1e^{ix_1}$ by a complex number $r_2e^{ix_2}$, the result is

$$r_1e^{ix_1}r_2e^{ix_2} = r_1r_2e^{i(x_1+x_2)}.$$

Hence, $r_1e^{ix_1}$ has been scaled by r_2 and rotated through an angle x_2. This is what makes complex multiplication so rich when compared with ordinary multiplication - it both scales *and* rotates complex numbers, when thought of as vectors in the plane.

Another incarnation of the complex numbers is as the following set of matrices.

$$M := \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

Exercise. *Prove that this set of matrices is closed under addition and multiplication.*

We say that this set of matrices M is **isomorphic** to the complex numbers because there exists a mapping $h : \mathbb{C} \to M$

$$h(a+bi) := \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

which provides a one-to-one correspondence between the complex numbers and the elements of M such that

$$h\Big((a+bi)+(c+di)\Big) = h(a+bi) + h(c+di)$$

and

$$h\Big((a+bi)(c+di)\Big) = h(a+bi) \times h(c+di).$$

This mapping h is called an **isomorphism**.

Exercise. *Prove the above assertions.*

Under this association the complex number re^{ix} corresponds to the matrix

$$r\begin{pmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{pmatrix}$$

This is the matrix which represents the linear transformation of $\mathbb{R}^2$ which rotates vectors in the plane anticlockwise by an angle of x, and then scales them by a factor of r. This provides a firm geometric picture of what complex multiplication *does*.

Earlier it was mentioned that the complex exponential satisfies $e^{i(x+y)} = e^{ix}e^{iy}$. Under the isomorphism h, this implies that we have the following equality of matrices:

$$\begin{pmatrix} \cos(x+y) & -\sin(x+y) \\ \sin(x+y) & \cos(x+y) \end{pmatrix} = \begin{pmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{pmatrix} \begin{pmatrix} \cos(y) & -\sin(y) \\ \sin(y) & \cos(y) \end{pmatrix}$$

By multiplying the matrices on the RHS, and then equating the entries with those of the matrix on the LHS, we obtain another proof of the trigonometric angle addition formulas.

Exercise. *Prove the above assertion.*

Another consequence of Euler's formula is that it allows us to give the following descriptions of sine and cosine:

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$
$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

where $x \in \mathbb{R}$.