Vectors

Definition. *We define* **n-*dimensional Euclidean space*** *to be the set*

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) : x_i \in \mathbb{R} \text{ for } 1 \le i \le n\}.$$

When $n = 1$ this is the real line $\mathbb{R}$, and when $n = 2$ this is the plane $\mathbb{R}^2$.

There are two basic operations we can perform on vectors. Suppose we have vectors $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$.

(i) The **addition** of $\mathbf{u}$ and $\mathbf{v}$ is defined component-wise

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \dots, u_n + u_n).$$

(ii) The **scalar multiplication** of $\mathbf{u}$ by $k \in \mathbb{R}$ is defined component-wise

$$k\mathbf{u} = (ku_1, ku_2, \dots, ku_n).$$

Definition. *Two vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be* ***parallel*** *if there exists $\lambda \in \mathbb{R}\setminus\{0\}$ such that $\mathbf{u} = \lambda\mathbf{v}$. In general, let $S = \{\mathbf{v_1}, ..., \mathbf{v_k}\}$ be a collection of vectors all from one fixed space, say $\mathbb{R}^n$. We say that S is* ***linearly independent*** *if for any $\lambda_1, ..., \lambda_k \in \mathbb{R}$ we have that*

$$\lambda_1\mathbf{v_1} + ... + \lambda_k\mathbf{v_k} = \mathbf{0} \text{ implies } \lambda_1 = \lambda_2 = ... = \lambda_k = 0.$$

Otherwise, we say that S is ***linearly dependent****. Note that two vectors are parallel iff they are linearly dependent.*

The **magnitude** of a vector $\mathbf{u} \in \mathbb{R}^n$ is defined to be

$$||\mathbf{u}|| = \sqrt{(u_1)^2 + (u_2)^2 + \cdots + (u_n)^2}.$$

A **unit vector** is a vector $\mathbf{u}$ such that $||\mathbf{u}|| = 1$. Given an arbitrary vector $\mathbf{u}$ we can form a unit vector in the same direction as $\mathbf{u}$ as follows

$$\hat{\mathbf{u}} = \frac{1}{||\mathbf{u}||}\mathbf{u}.$$

When working in $\mathbb{R}^3$ the unit vectors in the three coordinate directions have their own special notation. Namely,

$$\mathbf{i} = (1, 0, 0)$$
$$\mathbf{j} = (0, 1, 0)$$
$$\mathbf{k} = (0, 0, 1)$$

We can always resolve a vector into its coordinate components

$$\mathbf{u} = (u_1, u_2, u_3) = u_1(1, 0, 0) + u_2(0, 1, 0) + u_3(0, 0, 1) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}.$$

Definition. *We define the* ***distance*** *between two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ by*

$$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}|| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}.$$

Properties. *Suppose we have vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$*

(i) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$
(ii) $d(\mathbf{u}, \mathbf{v}) \ge 0$.
(iii) $d(\mathbf{u}, \mathbf{v}) = 0$ *iff* $\mathbf{u} = \mathbf{v}$.

Definition. *Suppose we have vectors $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$. The* ***dot product*** *of $\mathbf{u}$ and $\mathbf{v}$ is defined to be*

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

It gives a mapping $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

Properties. *Suppose we have vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $k \in \mathbb{R}$. The dot product has the following properties.*

(i) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
(ii) $k(\mathbf{u} \cdot \mathbf{v}) = k\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot k\mathbf{v}$
(iii) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
(iv) $\mathbf{u} \cdot \mathbf{u} = ||\mathbf{u}||^2 \ge 0$ *and* $\mathbf{u} \cdot \mathbf{u} = 0$ *iff* $\mathbf{u} = \mathbf{0}$.

The following theorem relates the magnitude of the dot product to the magnitude of the individual vectors.

Theorem (Cauchy-Schwarz). *Suppose we have vectors* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. *Then,*

$$|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}|| \, ||\mathbf{v}||$$

We have equality above iff $\mathbf{u}$ *is a multiple of* $\mathbf{v}$.

Theorem. *Suppose we have vectors* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. *Then,*

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| \, ||\mathbf{v}|| \cos(\theta)$$

where θ *is the angle between* $\mathbf{u}$ *and* $\mathbf{v}$.

When $n > 3$ we define the **angle** between two vectors to be the unique value of θ that makes the above identity true. Hence, this identity actually holds in arbitrary dimensions.

Corollary. *Suppose we have vectors* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. *These vectors are perpendicular iff* $\mathbf{u} \cdot \mathbf{v} = 0$.

Theorem (Pythagorean Theorem). *Suppose* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *are perpendicular vectors. Then*

$$||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2.$$

Theorem (Triangle Inequality). *Suppose we have vectors* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. *Then,*

$$||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||.$$

Definition. *Suppose we have vectors* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. *The* ***vector resolute*** *of* $\mathbf{u}$ *in the direction of* $\mathbf{v}$ *is defined to be* $(\mathbf{u} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}$. *The "signed length" of the vector resolute is* $\mathbf{u} \cdot \hat{\mathbf{v}}$ *and is called the* ***scalar resolute*** *of* $\mathbf{u}$ *in the direction of* $\mathbf{v}$.

We can decompose the original vector $\mathbf{u}$ into two components, one parallel to $\mathbf{v}$, and one perpendicular to $\mathbf{v}$ as follows

$$\mathbf{u} = (\mathbf{u} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}} + \Big(\mathbf{u} - (\mathbf{u} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}\Big).$$