Hyperbolic Functions

We define the hyperbolic functions as follows:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}.$$
$$\cosh(x) = \frac{e^x + e^{-x}}{2}.$$
$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}.$$

Each of the above functions has domain $\mathbb{R}$. You can think of $\cosh(x)$ as the average of e^x and e^{-x}, and $\sinh(x)$ as their average difference. This also helps to visualize their graphs. Just as the usual trigonometric functions $(\cos(t), \sin(t))$ parametrize the unit circle, in the same way the hyperbolic functions $(\cosh(t), \sinh(t))$ parametrize the right branch of the hyperbola $x^2 - y^2 = 1$. From the definitions we have the symmetries:

$$\sinh(-x) = -\sinh(x),$$
$$\cosh(-x) = \cosh(x),$$
$$\tanh(-x) = -\tanh(x).$$

We can also define the reciprocals of these functions:

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}.$$
$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}.$$
$$\coth(x) = \frac{1}{\tanh(x)}.$$

Note that:

$$e^x = \cosh(x) + \sinh(x) \quad \text{and} \quad e^{-x} = \cosh(x) - \sinh(x).$$

Hence, we immediately obtain the important identity:

$$\cosh^2(x) - \sinh^2(x) = 1.$$

Also, from the definitions we have:

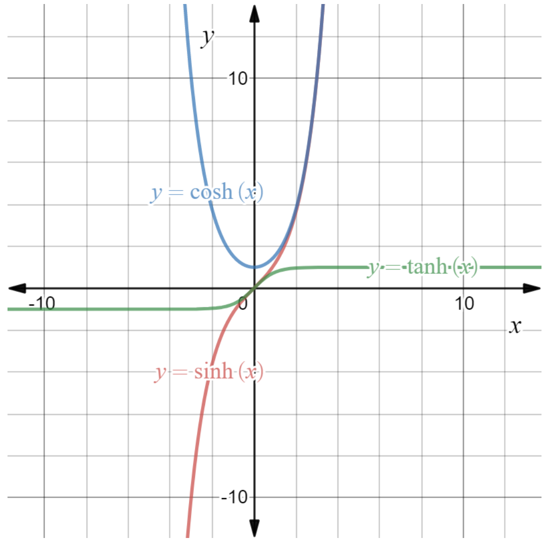
$$\sinh(x) = -i\sin(ix),$$
$$\cosh(x) = \cos(ix),$$
$$\tanh(x) = -i\tan(ix).$$

We can obtain many further identities analogous to those for the usual trigonometric functions. For example, the angle addition formulas for the hyperbolic functions are:

$$\sinh(x+y) = \sinh(x)\cosh(y) + \sinh(y)\cosh(x),$$
$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y),$$
$$\tanh(x+y) = \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)}.$$

The derivatives of the hyperbolic functions are as follows:

$\frac{d}{dx}\Big(\sinh(x)\Big) = \cosh(x)$	$\frac{d}{dx}\Big(\sinh^{-1}(x)\Big) = \frac{1}{\sqrt{x^2+1}}$
$\frac{d}{dx}\Big(\cosh(x)\Big) = \sinh(x)$	$\frac{d}{dx}\Big(\cosh^{-1}(x)\Big) = \frac{1}{\sqrt{x^2-1}}$ where $x > 1$
$\frac{d}{dx}\Big(\tanh(x)\Big) = \operatorname{sech}^2(x)$	$\frac{d}{dx}\Big(\tanh^{-1}(x)\Big) = \frac{1}{1-x^2}$ where $\lvert x\rvert < 1$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y = tanh (x)
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y = sinh (x)
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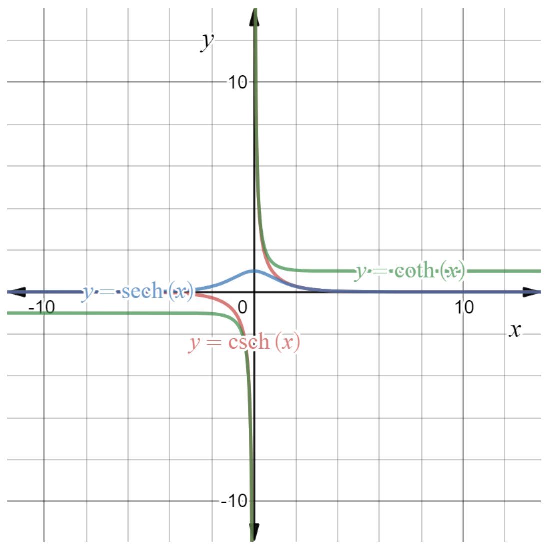
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y = coth (x)
y = sech (x)
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y = csch (x)
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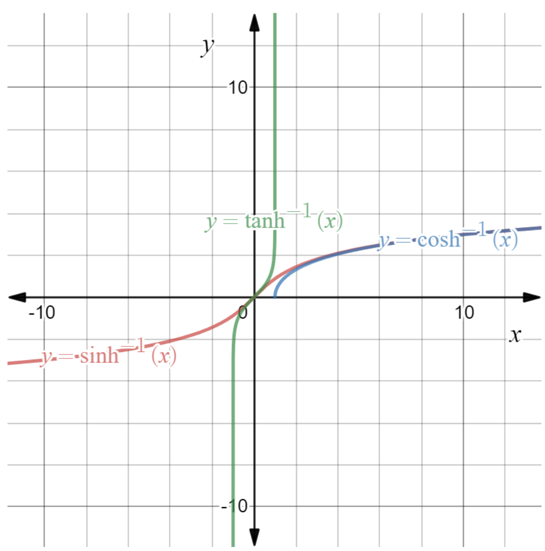
y
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y = tanh^−1 (x)
y = cosh^−1 (x)
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y = sinh^−1 (x)
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