Vector Products

Suppose that we have vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$. Below we will study two different "products" of these vectors. The **dot product** is a mapping $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ and the **cross product** is a mapping $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$.

The Dot Product. The dot product is a mapping $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined as follows

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Theorem. *Below is an alternative means of computing the dot product*

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||\,||\mathbf{b}|| \cos(\theta)$$

where θ is the angle between the two vectors.

From this characterization it is clear that the dot product is 0 iff the vectors are perpendicular. Intuitively, you should think of the dot product as providing a quantitative measure of the linear dependence of two vectors. It also appears when resolving vectors. For example, the scalar resolute of the vector $\mathbf{a}$ in the direction of $\mathbf{b}$ is $\mathbf{a} \cdot \hat{\mathbf{b}}$.

The following theorem relates the magnitude of the dot product to the magnitude of the individual vectors.

Theorem (Cauchy-Schwarz).

$$|\mathbf{a} \cdot \mathbf{b}| \leq ||\mathbf{a}||\,||\mathbf{b}||$$

We have equality above iff $\mathbf{a}$ *is a multiple of* $\mathbf{b}$.

The Cross Product. The cross product is a mapping $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ defined as follows

$$\mathbf{a} \times \mathbf{b} = \Big(a_2 b_3 - a_3 b_2\Big)\mathbf{i} - \Big(a_1 b_3 - a_3 b_1\Big)\mathbf{j} + \Big(a_1 b_2 - a_2 b_1\Big)\mathbf{k}.$$

Note that this definition implies that the cross product is anticommutative $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

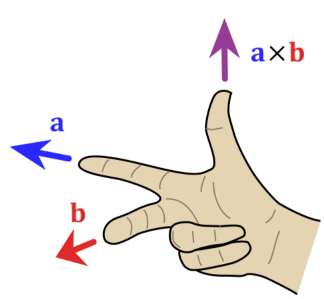
Theorem. *Below are alternative characterisations of the cross product.*

(i) $\mathbf{a} \times \mathbf{b}$ *is the vector whose magnitude is equal to the area of the parallelogram spanned by* $\mathbf{a}$ *and* $\mathbf{b}$*, and whose direction is perpendicular to both* $\mathbf{a}$ *and* $\mathbf{b}$*. In the equation below this direction is given by the unit vector* $\hat{\mathbf{n}}$*, and is uniquely specified by the right hand rule (see below). That is,*

$$\mathbf{a} \times \mathbf{b} = ||\mathbf{a}||\,||\mathbf{b}|| \sin(\theta)\hat{\mathbf{n}}.$$

(ii) $\mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$

The right hand rule is a convention for defining the direction of the cross product. It states that the direction is given by the direction of your thumb, when your right hand is aligned with the vectors as shown below.



Remark. *The following identities relate the cross-product and dot product.*

$$||\mathbf{a} \times \mathbf{b}||^2 + (\mathbf{a} \cdot \mathbf{b})^2 = ||\mathbf{a}||^2 ||\mathbf{b}||^2$$
$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

One More Product. If we consider three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$, then we can define another quantity of interest called the **triple product** (or box product) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. This product gives a mapping $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$. The triple product can be calculated as

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \pm \text{volume of the parallelepiped spanned by } \mathbf{a}, \mathbf{b}, \mathbf{c}.$$

Finally, note that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}).$$