Logic

Deductive reasoning is the process of reasoning from some collection of statements (the premises) to reach a conclusion. Provided that the premises are true, and the rules of logic are obeyed, the conclusion obtained must necessarily be true. In mathematics we abstract from the real world to an idealized context where we can establish results definitively using deductive reasoning. The extent to which our mathematical results are reflected in the real world depends upon the extent to which the axioms we assume in mathematics are true of the scenario under consideration.

The formal framework which we proceed to develop here is known as **Boolean logic** or **Boolean algebra**.

Definition. *Mathematical* ***statements*** *are assertions which are either true or false. We will usually denote statements by capital letters i.e.* P, Q.

There are four basic operations which we can perform on statements to obtain new statements. We describe these below.

Definition. *The* ***negation*** *of a statement* P*, denoted* $\neg P$ *and read "not* P*", is the statement that is true when* P *is false, and false when* P *is true.*

Definition. *The* ***conjunction*** *of two statements* P *and* Q*, denoted* $P \wedge Q$ *and read "*P *and* Q*", is the statement that is true when both* P ***and*** Q *are true, otherwise it is false.*

Definition. *The* ***disjunction*** *of two statements* P *and* Q*, denoted* $P \vee Q$ *and read "*P *or* Q*", is the statement that is true when either* P ***or*** Q *is true (or both are true), otherwise it is false.*

Definition. *Given two statements* P *and* Q*, we can form the new statement* $P \implies Q$ *read "*P ***implies*** Q*" or alternatively "If* P*, then* Q*". This statement is always true, unless* P *is true and* Q *is false, in which case it is false.*

We can represent each of the above operations using **truth tables**, in which T represents "true" and F represents "false".

P	$\neg P$
T	F
F	T

P	Q	$P \wedge Q$	$P \vee Q$	$P \implies Q$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

Note that given two statements P and Q, if $P \implies Q$ and $Q \implies P$, then we say that P and Q are **equivalent**, and write $P \Leftrightarrow Q$. We also say "P is true if and only if Q is true" or alternatively "P iff Q". From the truth table below is it clear that two statements are equivalent if they always have the same truth value.

P	Q	$P \implies Q$	$Q \implies P$	$P \Leftrightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

There are two operations we can perform on a statement involving an implication to obtain a new statement.

Definition. *The* ***converse*** *of the statement* $P \implies Q$ *is the statement* $Q \implies P$.

The converse statement is not logically equivalent to the original statement. This can be seen from the following truth table:

P	Q	$P \implies Q$	$Q \implies P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Definition. *The **contrapositive** of the statement $P \implies Q$ is the statement $\neg Q \implies \neg P$.*

The contrapositive statement is logically equivalent to the original statement. This can be seen from the following truth table:

P	Q	$P \implies Q$	$\neg Q \implies \neg P$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

For this reason we sometimes prove the contrapositive statement instead of the original statement, as it is sometimes easier to do so.

Theorem (De Morgan's Laws).

"not (P and Q)" is equivalent to "(not P) or (not Q)"
"not (P or Q)" is equivalent to "(not P) and (not Q)"

The theorem above can be proved by constructing an appropriate truth table.

Often when writing mathematical statements we employ the **logical quantifiers** "**for all**" denoted by $\forall$, and "**there exists**" denoted by $\exists$. It is crucial to understand the distinction between a property holding for all elements of a set, in contrast to there existing at least one element of a set for which the property holds. These symbols save us from writing out the words "for all" and "there exists" all the time. The order of these quantifiers in logical statements is very important! For example, what is the difference between the following two statements?

(i) $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}$ such that $y > x$.

(ii) $\exists y \in \mathbb{N}, \forall x \in \mathbb{N}$ such that $y > x$.