Quadratics

Definition. *A **quadratic** is an expression of the form*

$$ax^2 + bx + c$$

*where $a, b, c \in \mathbb{R}$. We refer to a, b, c as the **coefficients** of the quadratic. When $a = 1$ we say that the quadratic is **monic**.*

A quadratic is said to be **factorized** if it is written as a product of two linear factors

$$ax^2 + bx + c = a\Big(x^2 + \frac{b}{a}x + \frac{c}{a}\Big) = a(x - \alpha)(x - \beta).$$

It may be impossible to factorize a quadratic with $\alpha, \beta \in \mathbb{R}$.

Each quadratic has an associated function with rule

$$f(x) = ax^2 + bx + c.$$

A constant $r \in \mathbb{R}$ such that $f(r) = 0$ is referred to as a **root** or **zero** of the quadratic, or a **solution** of the equation $f(x) = 0$. A constant $r \in \mathbb{R}$ is a root of $ax^2 + bx + c$ iff $x - r$ is a factor of $ax^2 + bx + c$. Hence, a quadratic can have at most two roots.

We can also consider the graph of $f(x) = ax^2 + bx + c$. Every such graph has an **axis of symmetry** given by the line

$$x = -\frac{b}{2a}$$

and also a **turning point** that lies on this axis. When $a > 0$, the graph of the quadratic opens upward, and achieves its minimum at its turning point. When $a < 0$ the graph of the quadratic opens downward, and achieves its maximum at its turning point. The roots of the quadratic correspond to the x-intercepts of its graph. These can be explicitly determined in terms of the quadratic's coefficients. To do this we first **complete the square** to place the quadratic in **turning point form**:

$$a(x - h)^2 + k,$$

the coordinates of the turning point are then given by:

$$(h, k) = \Big(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\Big).$$

If k has the opposite sign to a, then we can factorize the quadratic as the difference of perfect squares to obtain an expression for the roots. This leads to the quadratic formula below.

Theorem. *The roots of the quadratic $ax^2 + bx + c$ are given by the **quadratic formula**:*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Hence,

$$ax^2 + bx + c = a\Big(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\Big)\Big(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\Big).$$

Note that the x-intercepts are symmetrically placed either side of the axis of symmetry.

We refer to the expression $b^2 - 4ac$ under the square root above as the **discriminant** of the quadratic. It provides information about the number of solutions of the quadratic.

Theorem. *Consider the quadratic $ax^2 + bx + c$.*
Suppose $a, b, c \in \mathbb{R}$. Then

- *(i) If $b^2 - 4ac > 0$, then the quadratic has two distinct real roots.*
- *(ii) If $b^2 - 4ac = 0$, then the quadratic has precisely one real root.*
- *(iii) If $b^2 - 4ac < 0$, then the quadratic has zero real roots.*

Importantly, a quadratic may have no real roots. This is manifested visually as such quadratics have graphs that lie entirely above, or entirely below, the x-axis.