Trigonometry

Trigonometry is the study of triangles. To this end the concept of an angle is fundamental. There are two common units used to measure angles: **degrees** and **radians**. Degrees are defined so that there are precisely 360° in a complete circle. Radians are defined so that there are precisely 2π radians in a complete circle.

Definition. *We define several important trigonometric functions as follows. Take the point* $(1, 0)$ *on the unit circle in the plane, and rotate it by* θ *degrees counter-clockwise about the origin. The* ***cosine*** *of* θ*, written* $\cos(\theta)$*, is defined to be the* x*-coordinate of the resulting point. The* ***sine*** *of* θ*, written* $\sin(\theta)$*, is defined to be the* y*-coordinate of the resulting point. The* ***tangent*** *of* θ*, written* $\tan(\theta)$*, is defined to be the ratio of the* y*-coordinate to the* x*-coordinate of the resulting point; this can also be thought of as the gradient of the unique line passing through the resulting point and the origin.*

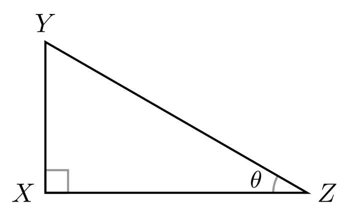
We also define the reciprocals of these functions the **cosecant**, **secant** and **cotangent**.

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

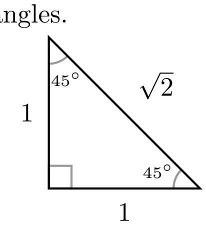
$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)}.$$

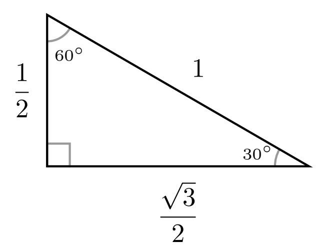
Trigonometric functions can also be introduced as the ratios of sides in a right triangle.



With respect to the triangle above we have $\sin(\theta) = \frac{XY}{YZ}$, $\cos(\theta) = \frac{XZ}{YZ}$, and $\tan(\theta) = \frac{XY}{XZ}$.

We highlight two specific triangles below. These allow us to compute the values of the trigonometric functions for various common angles.





θ	0	30°	45°	60°	90°	180°	270°	360°
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
$\tan(\theta)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	-	0	-	0

We list below a number of trigonometric identities.

$$\sin(-x) = -\sin(x)$$

$$\cos(-x) = \cos(x)$$

$$\tan(-x) = -\tan(x)$$

$$\cos^2(x) + \sin^2(x) = 1$$

$$\tan^2(x) + 1 = \sec^2(x)$$
$$\cot^2(x) + 1 = \csc^2(x)$$

$$\sin\Big(90^\circ + x\Big) = \cos(x)$$
$$\cos\Big(90^\circ + x\Big) = -\sin(x)$$
$$\tan\Big(90^\circ + x\Big) = -\cot(x)$$

$$\sin(180^\circ + x) = -\sin(x)$$
$$\cos(180^\circ + x) = -\cos(x)$$
$$\tan(180^\circ + x) = \tan(x)$$

$$\sin(x + y) = \sin(x)\cos(y) + \sin(y)\cos(x)$$
$$\sin(x - y) = \sin(x)\cos(y) - \sin(y)\cos(x)$$
$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$
$$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$
$$\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$$
$$\tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$$

$$\sin(2x) = 2\sin(x)\cos(x)$$
$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$$
$$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$$

$$\sin(x)\sin(y) = \frac{1}{2}\Big[\cos(x - y) - \cos(x + y)\Big]$$
$$\cos(x)\cos(y) = \frac{1}{2}\Big[\cos(x + y) + \cos(x - y)\Big]$$
$$\sin(x)\cos(y) = \frac{1}{2}\Big[\sin(x + y) + \sin(x - y)\Big]$$

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$
$$\sin(x) - \sin(y) = 2\sin\left(\frac{x - y}{2}\right)\cos\left(\frac{x + y}{2}\right)$$
$$\cos(x) + \cos(y) = 2\cos\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$
$$\cos(x) - \cos(y) = -2\sin\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right)$$

REMARK. *It is sometimes useful to combine linear combinations of trigonometric functions. Suppose $a, b \in \mathbb{R}$, then*

$$a\cos(x) + b\sin(x) = r\sin(x + \beta)$$

where $r = \sqrt{a^2 + b^2}$, $\sin(\beta) = \frac{a}{r}$, $\cos(\beta) = \frac{b}{r}$. The above identity relies on the angle addition formulas. By using such formulas we can similarly devise equality between the LHS and an expression of the form $r\cos(x + \alpha)$ for suitable choices of α.

Theorem. *Let ΔABC be any triangle. Let $s = \frac{1}{2}(a+b+c)$. The area $|\Delta ABC|$ of ΔABC is given by any of the following formulae.*

$$\begin{aligned}|\Delta ABC| &= \frac{1}{2}(base)(height) \\ &= \frac{1}{2}ab\sin(C) \\ &= \sqrt{s(s-a)(s-b)(s-c)}\end{aligned}$$

The following theorem is a generalization of the Pythagorean Theorem to triangles without a right angle.

Theorem (Law of Cosines). *Let ΔABC be any triangle. The following equality is satisfied*

$$c^2 = a^2 + b^2 - 2ab\cos(C).$$

Theorem (Law of Sines). *Let ΔABC be any triangle. The following equality is satisfied*

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}.$$

When first encountering trigonometry it is conceptually simpler to work in degrees, as we have done here. However, you will find that radians are a more common unit of measurement in most areas of mathematics.